TRINITY COLLEGE

Semester One Examination, 2023

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

If required by your examination administrator, please place your student identification label in this box

Section Two: Calculator-assumed

WA student number: In figures | | | | | | | | |

In words _____

_____

Your name _____

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	5	5	50	52	35
Section Two: Calculator-assumed	9	9	100	94	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

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Section Two: Calculator-assumed **65% (94 Marks)**

This section has **nine** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

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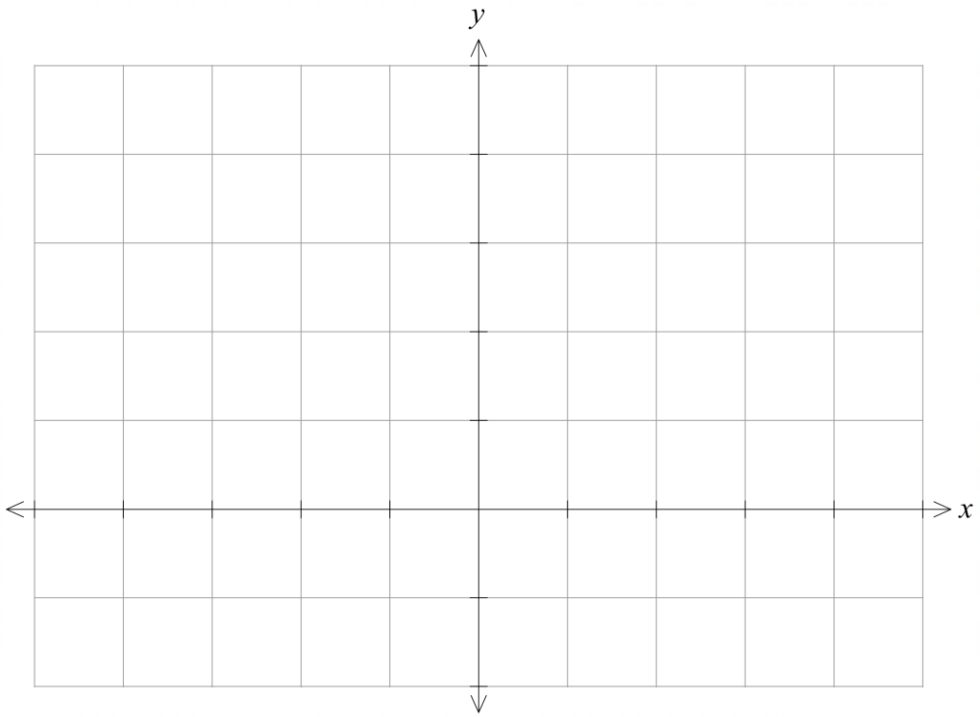
Question 6 **(5 marks)**

(a) Draw on the axes below the following functions. (3 marks)

$$y = |2x + 2|$$
$$y = |x| + 1$$



(b) Determine the coordinates of the point(s) where the graphs intersect. (2 marks)

See next page

Question 7 **(6 marks)**

Two planes have equations

$$\Pi_1: x + y + z = 6$$
$$\Pi_2: x + y + 2z = 9$$

Three further planes have equations

$$\Pi_A: 2x + 2y + 3z = 15$$
$$\Pi_B: x + 2y + 3z = 14$$
$$\Pi_C: 2x + 2y + 2z = 15$$

Consider planes Π_1 and Π_2. Determine which of the three planes Π_A, Π_B or Π_C, together with Π_1 and Π_2,

(a) intersect at a single point, and determine the coordinates of this point. (2 marks)

(b) intersection along a line, and determine the vector equation of that line. (4 marks)

See next page

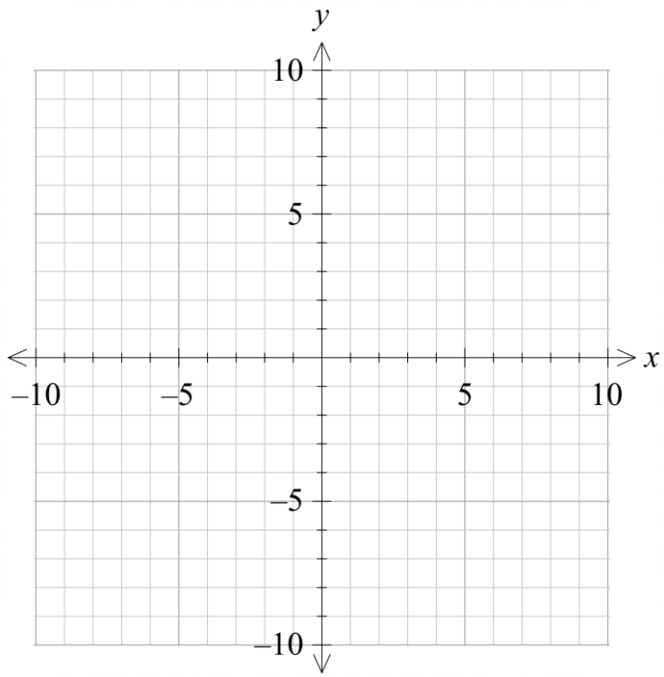
Question 8 **(9 marks)**

(a) Express $x^2 - 8x + 7$ in the form $(x + b)^2 + c$. (1 mark)

(b) Explain why the domain of $f(x) = x^2 - 8x + 7$, must be restricted for $f^{-1}(x)$ to be a function. (2 marks)

(c) Determine the largest domain of $f(x)$, such that $f^{-1}(x)$ is in the first and second quadrants. (2 marks)

(d) On the axes below sketch $f(x)$ over the domain identified in part (c), as well as the graph of $f^{-1}(x)$. (4 marks)



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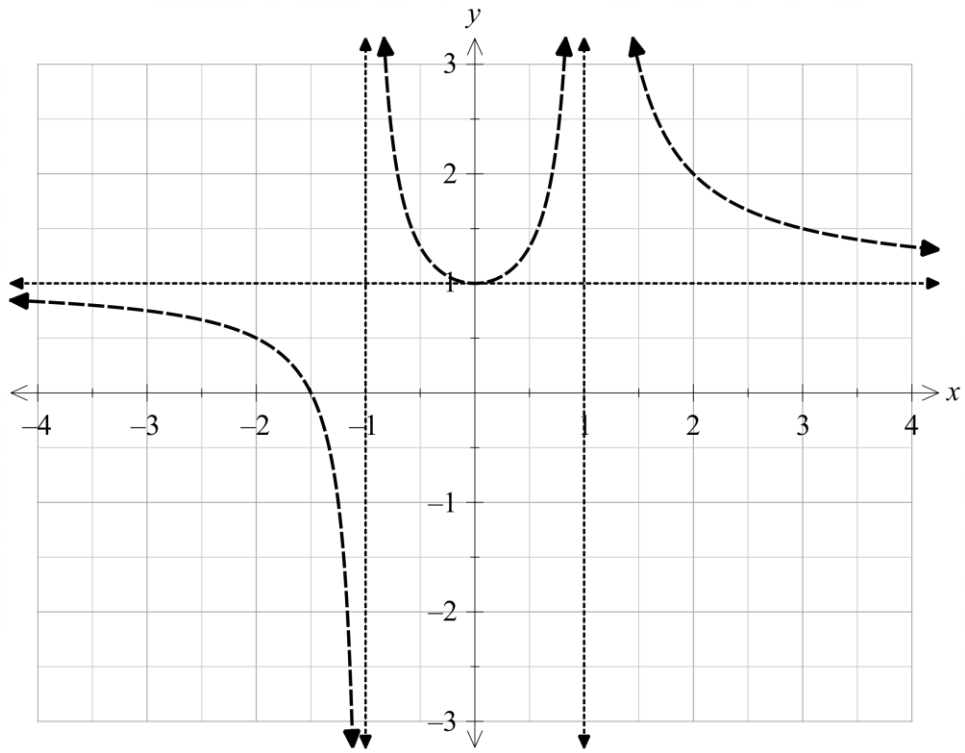
Question 9 **(9 marks)**

The graph of $y = \frac{1}{f(x)}$ is shown with a dotted curve on the axes below.



(a) On the same axes draw the graph of $f(x)$. (4 marks)

(b) (i) The equation $|f(x)| = k$ has 4 solutions for what range of values of k? (2 marks)

(ii) Does the equation $|f(x)| = k$ ever have 3 solutions? (1 marks)

(c) Determine the solutions to $f(-|x|) = 2$. (2 marks)

See next page

Question 10 **(9 marks)**

(a) Prove the following identity (3 marks)

$$z + \frac{|z|^2}{z} = 2\text{Re}(z)$$

(b) Let $\mathbf{u} = \langle 3, -1, 5 \rangle$ and $\mathbf{v} = \langle 0, a, b \rangle$. Given that $\mathbf{v}$ is a unit vector perpendicular to $\mathbf{u}$, find the possible value(s) of a and b. (3 marks)

See next page

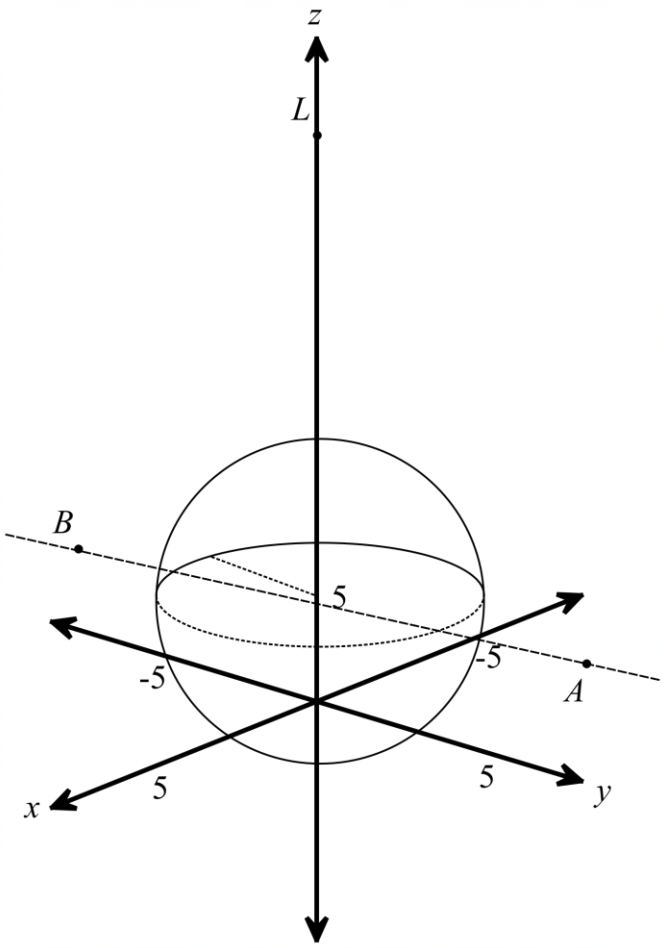
(c) Given that $\mathbf{a} \times \mathbf{b} = -2(\mathbf{b} \times \mathbf{c})$, prove that $\mathbf{a} - 2\mathbf{c}$ is a scalar multiple of $\mathbf{b}$. (3 marks)

See next page

Question 11 **(13 marks)**

The diagram at right shows the following:

- a line passing through the points $A\ (-1,7,4)$ and $B\ (7,1,6)$,
- a sphere with centre $C\ (0,0,5)$ that is tangent to the xy-plane.
- a point at $L\ (0,0,15)$



(a) Determine the vector equation of the sphere. (2 marks)

A number of lights are located in a plane containing the point L, that is parallel to the xy-plane. These lights shine down directly on the line and sphere and cast shadows on the xy plane. You may assume that the light rays are all parallel to the z axis.

The Cartesian equation of the shadow of the sphere is $x^2 + y^2 = 25$.

(b) Determine the Cartesian equation of the shadow of the line. (3 marks)

See next page

(c) Clearly show that the shadow of the line touches the shadow of the sphere in the xy plane. (3 marks)

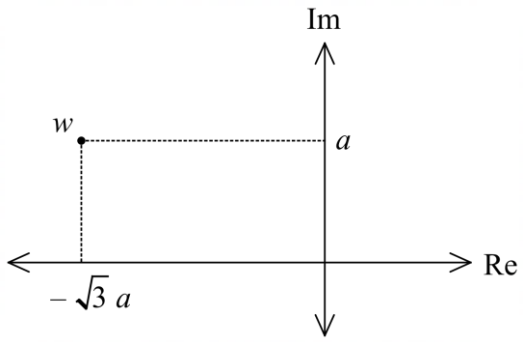
(d) (i) Explain why part (c) does not prove that the line passing though A and B touches the sphere. (1 mark)

(ii) Determine if the line passing though A and B touches the sphere. If it does determine the point(s) of intersection. If it does not, explain why. (4 marks)

See next page

Question 12 **(15 marks)**

The complex number w has been plotted on the Argand diagram below.



(a) Express w in Cartesian form. (1 mark)

(b) Express w in polar form. (3 marks)

(c) The complex number z_1 is a root of $z^6 = w$, with the smallest positive argument.

(i) Given that $a = 32$, determine z_1 in polar form. (3 marks)

(ii) Determine the remaining roots in polar form. Label the roots as z_2, z_3, z_4, z_5 and z_6 moving in an anticlockwise direction from the positive real axis. (2 marks)

See next page

(d) Determine the exact polar form of $z_1 + z_2$. (3 marks)

$z_1 + z_2$, $z_3 + z_4$ and $z_5 + z_6$ are roots of $z^3 = k\ cis\left(\frac{11\pi}{m}\right)$.

(d) Determine the values of k and m. (3 marks)

See next page

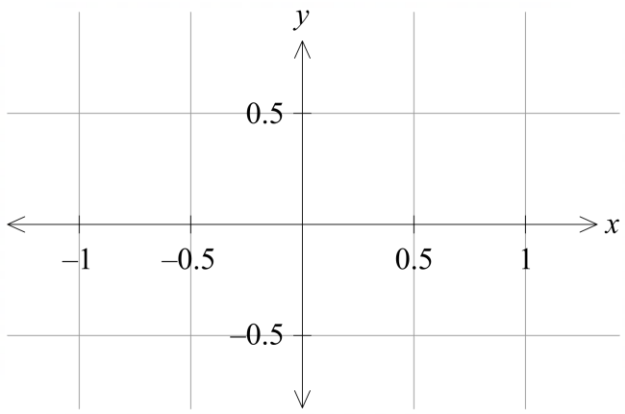
Question 13 **(16 marks)**

A carnival ride is moving along the curve defined parametrically as

$$\begin{cases} x(t) = \sin t \\ y(t) = \dfrac{\cos t}{2 + \sin t} \end{cases} \quad 0 \leq t \leq 2\pi$$

where x and y are measured in metres, and t in minutes.

(a) Sketch the graph of this curve on the axes below. (2 marks)



(b) Show that the Cartesian equation of the curve is given by (2 marks)

$$y^2 = \frac{1 - x^2}{4 + 4x + x^2}$$

(c) Indicate on the curve drawn in part (a),

(i) the starting position of the ride with an S. (1 mark)

(ii) the direction of motion of the ride with an arrow. (1 mark)

See next page

(d) Determine the initial speed of the ride. (3 marks)

(e) Determine the position of ride when it is moving most slowly. (3 marks)

The ride has a camera that starts from rest at $(0, 0.5)$ and travels to $(0, -0.5)$ where it stops after 3 minutes.

The acceleration of the camera is given by $\mathbf{a} = \langle 0, 0.2t - 0.3 \rangle$.

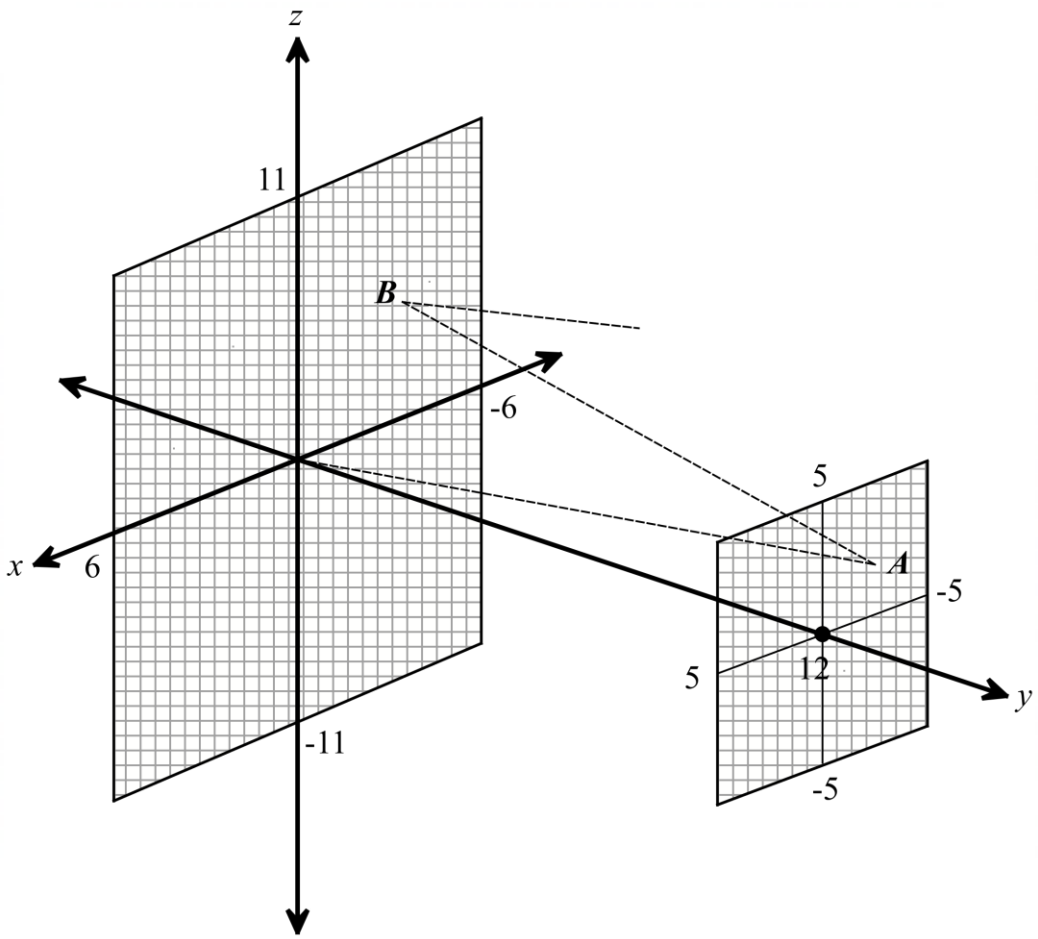
(f) Determine the velocity vector of the camera. (2 marks)

(g) Determine the displacement vector of the camera. (2 marks)

See next page

Question 14 **(12 marks)**

Two parallel mirrors are shown in the diagram below. The larger mirror passes through the origin and is coincident with the xz plane, and the smaller mirror is in the plane $y = 12$.



A laser beam is fired through a small hole at the origin. The dotted line shows one such beam. The beam then hits the mirror at $y = 12$ and is reflected back towards the larger mirror.

The laser beam is pointed with direction $\mathbf{d} = -\mathbf{i} + 6\mathbf{j} + \mathbf{k}$.

(a) Determine the position vector of, A, the point where the beam hits the smaller mirror.

(4 marks)

See next page

The laser beam is then reflected with direction $\mathbf{d} = -\mathbf{i} - 6\mathbf{j} + \mathbf{k}$.

(b) Determine the position vector of, B, the point where the beam hits the larger mirror.

(3 marks)

A second beam is fired from the origin with a direction of $\mathbf{d}_1 = a\mathbf{i} + 6\mathbf{j} + c\mathbf{k}$. When it hits the smaller mirror, it is then reflected with direction of $\mathbf{d}_2 = a\mathbf{i} - 6\mathbf{j} + c\mathbf{k}$. You may assume that the speed of the beam does not change.

There are laser beams from the origin which after being reflected in the small mirror do not hit the larger mirror.

(c) Determine the range of values of a and c, that ensure the beams **are reflected** in the larger mirror. (5 marks)

End of questions

Supplementary page

Question number: _____

Supplementary page

Question number: _____

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